Linear Algebra 1

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<u>Informal Def</u> A vector space V over "F" is a set of "vectors" $\vec{v} \in V$ s.t.

1. You can add vectors: $\vec{v} + \vec{w} \in V$
2. You can scale vectors by elements of F: $r\vec{v} \in V \ \forall r \in F$

satisfying axioms (Ex: can subtract vectors)

<u>Ex 1:</u> $V = \mathbb{R}^n$, $F = \mathbb{R}$ $\vec{v} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

- $\vec{v} + \vec{w}$ = add componentwise
- $r\vec{v} = \begin{pmatrix} ra_1 \\ \vdots \\ ra_n \end{pmatrix} \quad \forall r \in \mathbb{R}$

<u>Ex 2:</u> $V = \mathbb{R}|dog\rangle \oplus \mathbb{R}|cat\rangle$, $F = \mathbb{R}$

- $\vec{v} = a|dog\rangle + b|cat\rangle$, $\vec{w} = c|dog\rangle + d|cat\rangle$

$\vec{v} + \vec{w} = (a+c)|dog\rangle + (b+d)|cat\rangle$

- $r\vec{v} = ra|dog\rangle + rb|dog\rangle \ \forall r \in \mathbb{R}$

<u>Ex 3:</u> $V = \mathbb{R}[x_1, \ldots, x_n] :=$ polynomials in n-variables, $F = \mathbb{R}$

- $\vec{v} = f(x_1, \ldots, x_n)$, $\vec{w} = g(x_1, \ldots, x_n)$

  $\vec{v} + \vec{w} = f(x_1, \ldots, x_n) + g(x_1, \ldots, x_n)$
- $r\vec{v} = r(f(x_1, \ldots, x_n))$

<u>Warning</u>: Here V is infinite-dim unlike in Ex 1, 2.

<u>Def</u>: A map $T: V \to U$ is linear if

(a) $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$, $\vec{v}, \vec{w} \in V$

(b) $T(c\vec{v}) = cT(\vec{v})$, $c \in F$

<u>Ex 4:</u> $V = \mathbb{R}[x_1, \ldots, x_n]$, $T = \frac{\partial}{\partial x_i}$ is linear

$\frac{\partial}{\partial x_i}(f+g) = \frac{\partial}{\partial x_i} f + \frac{\partial}{\partial x_i} g$, $\frac{\partial}{\partial x_i}(cf) = c\frac{\partial}{\partial x_i} f$

Linear Algebra 2

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Def A subset $B \subseteq V$ is a basis if

(1) For every finite subset $\{\vec{v}_1, \dots, \vec{v}_r\}$ of B

if $c_1\vec{v}_1 + \dots + c_m\vec{v}_m = 0$ for some $c_1, \dots, c_m \in F$

then $c_1 = \dots = c_m = 0$ (L.I.)

(2) $\forall \vec{v} \in V$, $\exists\ a_1, \dots, a_n \in F$, $\exists\ \vec{v}_1, \dots, \vec{v}_n \in B$

s.t. $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$ (spanning)

Rem (1) implies that $\{a_i\}$ are unique.

(2) If V is f.d. ($|B|$ is finite) only need to check (1) for B

Lemma 1: By fixing a basis $B_n = \{\vec{v}_1, \dots, \vec{v}_n\}$ of $\mathbb{R}^n$ and a basis

$B_m = \{w_1, \dots, w_m\}$ of $\mathbb{R}^m$

$$\left\{\begin{array}{c}\text{linear maps} \\ \mathbb{R}^n \to \mathbb{R}^m\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c} m\times n \text{ matrices} \\ M_{m\times n}(\mathbb{R})\end{array}\right\}$$

$$T \longleftrightarrow M_T = \begin{array}{cc} T(v_1) \ \dots & T(v_n) \end{array}$$
$$M_T = \begin{pmatrix} a_{11} & & a_{1n} \\ a_{m1} & \ddots & a_{mn} \end{pmatrix}, \quad M_T\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} = \begin{pmatrix}a_{11}\\\vdots\\a_{m1}\end{pmatrix} = T\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$$

$T(\vec{v}_1) = a_{11}\vec{w}_1 + \dots + a_{m1}\vec{w}_m$

$\vdots$

$T(v_n) = a_{1n}\vec{w}_1 + \dots + a_{mn}\vec{w}_n$

Ex 5: $V = \mathbb{R}^n$, then $B = \left\{ \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\\vdots\\1\end{pmatrix} \right\}$ is a basis for V

Question 1: Is $B = \left\{ \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix} \right\}$ a basis for $\mathbb{R}^2$?

A: Yes $\det(M_{T_B}) = 5 \neq 0$

Lemma: If T is an invertible linear map ($\exists T^{-1}$ linear s.t. $T^{-1}T = TT^{-1} = I$) then

$T(\vec{v}) = 0 \Rightarrow \vec{v} = 0$

Pf: $\vec{v} = T^{-1}(T(\vec{v})) = T^{-1}(0) = 0$

Linear Algebra 3

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Prop 2 If $\{v_1, \dots, v_n\}$ is a basis for V, $T: V \to V$ invertible linear map, then $\{T(v_1), \dots, T(v_n)\}$ is also a basis for V.

Pf: (1) Suppose $\sum_{i=1}^{n} c_i T(v_i) = 0$

$$\overset{(b)}{=} \sum_{i=1}^{n} T(c_i v_i) \overset{(a)}{=} T\left(\sum_{i=1}^{n} c_i v_i\right) = 0$$

$$\overset{\text{Lem}}{\Longrightarrow} \sum_{i=1}^{n} c_i v_i = 0 \overset{\text{L.I.}}{\Longrightarrow} c_i = 0 \ \forall i$$

(2) B/c $\{v_i\}$ is a basis $T^{-1}(w) \overset{\text{span}}{=} \sum_{i=1}^{n} a_i v_i$ $\forall w \in V$

$$\Rightarrow w = T(T^{-1}(w)) = T\left(\sum a_i v_i\right)$$

$$\overset{(a)}{=} \sum_{i=1}^{n} T(a_i v_i) \overset{(b)}{=} \sum_{i=1}^{n} a_i T(v_i)$$

Question 2 How to tell if T is invertible

A: T is invertible $\Longleftrightarrow \det M_T \neq 0$

Back to Question 1: Define $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$, $T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}$

$\Rightarrow M_T = \begin{pmatrix}2 & 3\\1 & 4\end{pmatrix}$, $\det M_T = 8 - 3 = 5$

$\Rightarrow M_T$ invertible $\Rightarrow \left\{\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^2$. In general,

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Thrm 3: let $\{w_i\} = \{v_i\}_{i=1}^n$ in lemma 1 be a basis for V. Then $\{z_i\}_{i=1}^n$ is a basis for $V \Longleftrightarrow \det(M_{T_z}) \neq 0$ where

$$M_{T_z} = \begin{pmatrix} a_{11} & & a_{1n} \\ & & \\ a_{n1} & & a_{nn} \end{pmatrix} \qquad \begin{aligned} z_1 &= a_{11} v_1 + \dots + a_{n1} v_n \\ &\ \vdots \\ z_n &= a_{1n} v_1 + \dots + a_{nn} v_n \end{aligned}$$

Pf: Let $T_z(v_i) = z_i$ in Prop 2. Then use Q2 + Lem 1.

Ex 6: $\{1, x, x^2\}$ is a basis for $P = \text{Poly}_{\leq 2}$ ← poly of at most deg 2

Is $\{1, x+1, (x+1)^2\}$ a basis for P?

Permutations

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A: $1=1$, $x+1 = 1\cdot x+1$, $(x+1)^2 = 1\cdot x^2 + 2\cdot x + 1$

$\rightsquigarrow M_{T_2} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$, $\det M_{T_2} = 1 \neq 0$

So Yes!

<u>Def</u> An inner product on a v.s. $V/\mathbb{R}$ is a map $\langle , \rangle : V\times V \to F$ s.t. $\forall u,w,u \in V$ $a,b \in \mathbb{R}$

(1) $\langle av+bw, u\rangle = a\langle v,u\rangle + b\langle w,u\rangle$

(2) $\langle v,w\rangle = \langle w,v\rangle$

(3) $\langle v,v\rangle > 0 \quad \forall v \neq 0 \in V$

<u>Ex 6</u> $V = \mathbb{R}|dog\rangle \oplus \mathbb{R}|cat\rangle$. Let

$\langle |dog\rangle, |dog\rangle\rangle = 2 \quad \langle |cat\rangle, |cat\rangle\rangle = 1$

$\langle |dog\rangle, |cat\rangle\rangle = \langle |cat\rangle, |dog\rangle\rangle = 0$

Then extend by linearity, aka impose (1)

(2) ✓ (3) ✓

- In general, to define $\langle , \rangle$, define $\langle v_i, v_j\rangle$ $\forall v_i, v_j$ of a basis for V and check (2), (3)

<u>Permutations</u>: Let $[n] = \{1,2,\dots,n\}$

$S_n = \{f : [n] \to [n] \mid f \text{ is a bijection}\}$

Different ways to represent ele of S_n

<u>Two line notation</u>:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \in S_4$$

<u>cycle notation</u>: $\pi = (1\,2\,4\,3) \in S_4$

- $(\dots ab \dots)$ means a goes to b

<u>To convert π to cycle notation</u>: start with 1, compute $\pi(1)$, then $\pi(\pi(1))$, ...

$\rightsquigarrow (1 \;\pi(1)\; \pi(\pi(1)) \dots$ until $\pi^k(1) = 1$ then "close" cycle.

Permutations 2

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- start again with next lowest number not in cycle. Continue until all numbers are covered

Exer: Convert π to cycle notation

1. $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}$ 2. $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 5 & 2 \end{pmatrix}$

Rem: If you have fixed points (x), pp'l omit this, alca the answer to 2 will be just (1452)

Symmetric Polynomials

Def: $f(x_1, \dots, x_n) \in \mathbb{K}[x_1, \dots, x_n]$ is symmetric if $\pi f = f \; \forall \pi \in S_n$ where

$$\pi f = f(x_{\pi(1)}, \dots, x_{\pi(n)})$$

Let $\text{Sym}_n = \Lambda(X_n) =$ symmetric poly in $x_1, \dots, x_n$

- Let $f(X_n) := f(x_1, \dots, x_n)$

Prop Let $c \in \mathbb{K}$, $\pi, \theta \in S_n$. Then

(i) $\pi(cf) = c\pi(f)$

(ii) $\pi(f+g) = \pi(f) + \pi(g)$

(iii) $\pi(fg) = \pi(f)\pi(g)$

(iv) $(\pi\theta)(f) = \pi(\theta(f))$

Note if $f \in \Lambda(X_n)$, $\pi(cf) = c\pi(f) = cf$
$\pi(f+g) = f+g$

Cor (a) $\Lambda(X_n)$ is a v.s.

(b) $f, g \in \Lambda(X_n) \Rightarrow fg \in \Lambda(X_n)$

(c) If $(i\; i+1)(f) = f \;\; \forall\, 1 \le i \le n-1$ then $f \in \Lambda(X_n)$

Pf: (c) $(1\,2), \dots, (n-1\;n)$ "generate" S_n

Ex 7: $n=3$, $f = x_1^2 x_2 x_3 + x_2^2 x_1 x_3 + x_3^2 x_1 x_2$
$\in \Lambda(X_3)$. $(12)f =$ $(23)f =$

Symmetric Functions

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Def symmetric function $= Sym_\infty = \Lambda(X)$

aka a symmetric poly in ∞-many variables

Question 3: Why ∞-many variables?

A: We want to study algebraic properties w/o special cases.

Ex: "Informal def"

$p_1(X_1) = e_1(X_1) = x_1$

$p_1(X_2) = e_1(X_2) = x_1 + x_2$

$e_1(X_3) = x_1 + x_2 + x_3$
$\quad \| $
$p_1(X_3)$ $\vdots$

$e_2(X_1) = 0$

$e_2(X_2) = x_1 x_2$

$e_2(X_3) = x_1x_2 + x_1x_3 + x_2x_3$

$\vdots$

$p_2(X_1) = x_1^2$

$p_2(X_2) = x_1^2 + x_2^2$

$p_2(X_3) = x_1^2 + x_2^2 + x_3^2$

$\vdots$

$p_3(X_1) = x_1^3$

$p_3(X_2) = x_1^3 + x_2^3$

$p_3(X_3) = x_1^3 + x_2^3 + x_3^3$

$\vdots$

Question 4: What is $e_2e_1(X_k)$ in terms of $p_i(X_k)$?

A: • $e_1e_2(X_2) = x_1^2x_2 + x_1x_2^2$

Now, $p_2p_1(X_2) = (x_1^2 + x_2^2)(x_1 + x_2)$
$= x_1^3 + x_1^2x_2 + x_2^2x_1 + x_2^3$

$\Rightarrow e_1e_2(X_2) = p_2p_1(X_2) - p_3(X_2)$

• $e_1e_2(X_3) = x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3$
$+ x_3^2x_1 + x_3^2x_2 + 3x_1x_2x_3$
$= p_2p_1(X_3) - p_3(X_3) + 3e_3(X_3)$

Rem: $\forall k \geq 3$ we have the identity

$e_1e_2(X_k) = p_2p_1(X_k) - p_3(X_k) + 3e_3(X_k) \quad (*)$

$\rightsquigarrow$ In $\Lambda(X)$

$e_1e_2 = p_2p_1 - p_3 + 3e_3$

Symmetric Functions 2

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- In general, mult of sym poly always stablizes as $n \to \infty$.

Def: Let $\Lambda_k(X_n)$, $\Lambda_k(X)$ = sym poly (functions) homogeneous of deg k (all terms deg k)

Prop: To prove a homogenous identity of deg k in $\Lambda(X)$, sufices to prove it in $\Lambda_k(X_n)$, $n \geq k$

Rem: Also for $n=k$ will give limiting behavior of your identity. $(*)$ was homogeneous of deg 3, so $k=3$ captured long term behavior.

Algebra	Combinatorics
• $\Lambda(X)$	• $\Lambda(X_n)$
• easier to prove things	• easier to compute things

Partitions

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Def Let $n \in \mathbb{Z}^{>0}$. A partition λ of n is a weakly decreasing sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \qquad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$$

s.t. $\sum_{i=1}^{r} \lambda_i = n$. Write $\lambda \vdash n$, $|\lambda| = n$

$$\ell(\lambda) = r = \#\text{ of parts of } \lambda$$

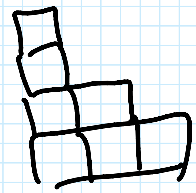
Ex 8: $(3,2,1,1) \vdash 7$

Rem: If integers are repeated, ppl sometimes consolidate notation by using exponents, e.g.

$$(3,2,1,1) = (3,2,1^2)$$

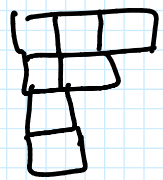
$$\underbrace{(1, \ldots, 1)}_{k \text{ times}} = 1^k$$

Def Given $\lambda \vdash n$, the (French) Young diagram = stack λ_1 boxes on bottom, stack λ_2 boxes above, etc



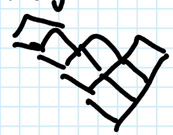
$\longleftrightarrow \lambda = (3,2,1,1)$

Def Given $\lambda \vdash n$ the (English) Young diagram = stack λ_1 boxes on top, stack λ_2 boxes below, etc



$\longleftrightarrow \lambda = (3,2,1,1)$

Def Given $\lambda \vdash n$ the (Russian) Young diagram = rotate French by $45°$



$\longleftrightarrow \lambda = (3,2,1,1)$